A Comparison of the Berenger Perfectly Matched Layer and the Lindman Higher-Order ABC's for the FDTD Method

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Abstract— **Higher-order absorbing boundary conditions are compared to the recently introduced Berenger perfectly matched layer (PML) absorbing boundary conditions (ABC). Reflections caused by the ABC's are examined in both the time and frequency domains for the case of a line source radiating in a finite computational domain. It is shown that the PML ABC significantly reduces reflections from the truncation of the computational grid when compared to 7th order Lindman ABC's. Also, except for at low frequencies, higher-order absorbing boundary conditions are no better than 2nd order Mur absorbing boundaries.**

I. Introduction

THE Finite-Difference Time-Domain (FDTD) method for the analysis of electromagnetic interaction is a powerful technique which has found widespread use in recent years. The method as used for electromagnetic scattering, radiation and propagation problems was first introduced by Yee [1]. Yee proposed the discretization of the differential form of Maxwell's equations in time and space using second order accurate central differences. The resulting difference equations are then solved in a time marching sequence by alternately calculating the electric and magnetic fields on an interlaced Cartesian grid. The FDTD method is being applied to many problems involving scattering or radiation in open domains. The solution of these problems requires the use of radiation or absorbing boundary conditions to accurately terminate the computational domain allowing the propagation of electromagnetic energy out of the computational space.

Analytical absorbing boundary conditions (ABC's) developed for open problems include one-way wave equation (OWWE) based methods. These ABC's are obtained by starting with the wave equation and deriving OWWE's that allow propagation in the outward direction. Lindman [2] was the first to propose the use of OWWE's at the truncation of the computational space. He suggested the replacement of the radical in the resulting OWWE with a series approximation. The Enquist and Majda [3] approach, of which Mur [4] provided the discretization and application to the Yee algorithm, is the most widely applied and uses Padé polynomials to approximate the radical. Halpern and Trefethen [5] proposed several classes of approximations to the radical involved including least squares, minmax, Chebyshev and Chebyshev-Padé. Renaut [6] showed that the extension of these methods to higher orders is neither obvious nor unique and that their stability cannot be assumed. She also observed that any approximation to the radical can be presented in a form equivalent to that proposed by Lindman. Tirkas, Balanis and Renaut [7] demonstrated the accuracy and efficiency of the Halpern and Trefethen approach and the Lindman approach as extended by Renaut. They showed that the higher-ordered Lindman ABC (HOABC) with standard Padé coefficients exhibited the best performance for the TM_z case in a computational domain of 2 : 1 dimension. Subsequently, McInturff and Simon [8] provided closed form expressions for the Lindman expansion coefficients used in these higher-order ABC's.

Recently a new ABC has been introduced by Berenger [9], extended and validated by Katz *et al.* [10] and applied by Reuter *et al.* [11]. The Berenger "Perfectly Matched Layer" (PML) involves the application of a nonphysical absorbing material adjacent to the outer computational boundary. The method is implemented by splitting certain field components in the PML region into subcomponents. The PML material has characteristics which permit electromagnetic waves of arbitrary frequency and angle of incidence to be absorbed while maintaining the impedance and velocity of a lossless dielectric. Berenger reported reflection coefficients for PML in two dimensions significantly better than second and third order OWWE based ABC's. Katz *et al.* extended the PML method to three dimensions and verified their effectiveness in the time domain versus second order Mur ABC's. It is the purpose of this paper to compare the time and frequency domain effectiveness of the Berenger PML ABC, the HOABC and Mur's 2nd order ABC.

II. Numerical Experiments

In this section, numerical simulations are conducted to compare the effectiveness of the Berenger PML, the HOABC of Tirkas, Balanis and Renaut and the Mur 2nd order ABC. The methodology for the time domain experiments is the same as that published in [12] and used in most papers introducing and comparing ABC's. The procedure involves exciting a z-directed electric line source centered within the vacuum region of a 100 × 50 cell test grid, Ω_T. A square Yee cell is used

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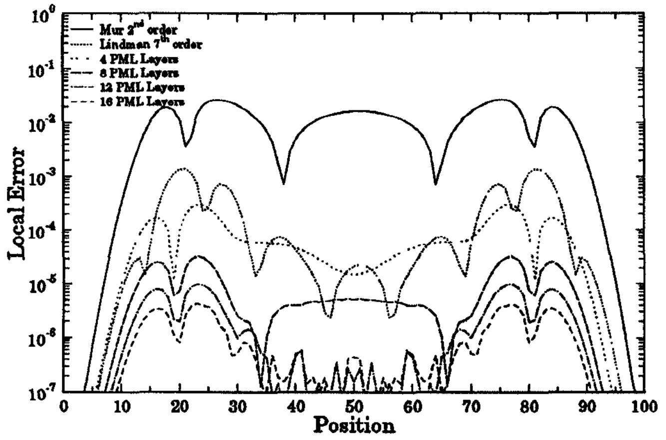

Fig. 1. Local error induced by various ABC's on the boundary after 100 time steps.

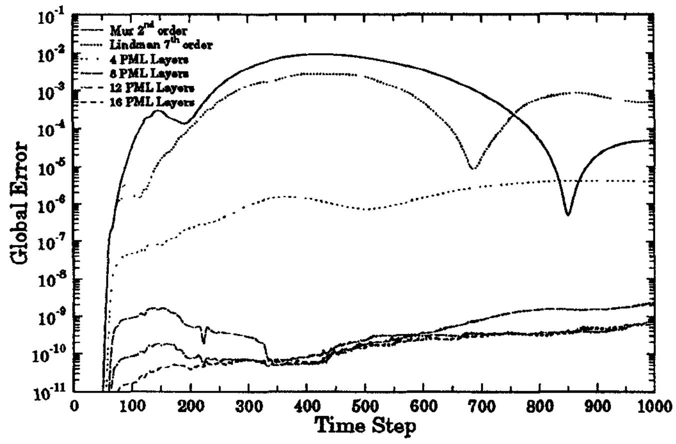

Fig. 2. Global error induced in the test domain by various ABC's as a function of time.

with $\Delta x = \Delta y = 0.015$ m. The time step, Δt, is set using $\Delta t = \Delta x/2c = 2.5\times10^{-11}$ s where c is the speed of light in a vacuum. At the upper end of the frequency band of interest, 3 GHz, $\Delta x = \lambda/6.667$ and the electrical distance to the nearest absorbing boundary is $25\Delta x$ or 3.75λ. Toward the lower end of the band at 100 MHz, $\Delta x = \lambda/200$ and the electrical distance to the nearest absorbing boundary is 0.125λ.

The excitation is a pulse having very smooth transition to zero as used in [12] and defined as follows:

$$E_z(50,25,n) = \begin{cases} \alpha(10 - 15\cos\omega_1\xi + 6\cos\omega_2\xi - \cos\omega_3\xi) & \xi \le \tau \\ 0 & \xi > \tau \end{cases} \tag{1}$$

where

$$\alpha = \frac{1}{320}, \qquad \tau = 10^{-9}, \quad \xi = n\Delta t,$$

$$\omega_m = \frac{2\pi m}{\tau}, \quad m = 1,2,3.$$

The pulse has significant low frequency and dc components where OWWE based ABC's are widely recognized as being inaccurate. The test grid, Ω_T, is terminated by either a 2nd order Mur ABC, a 7th order Lindman ABC or a PML with a reflection coefficient of 10^{-5} backed by PEC walls. A control FDTD solution having no errors due to reflections from an ABC is obtained by running a large FDTD domain, Ω_B, centered on Ω_T and having an outer boundary so remote that reflections from its grid truncation are isolated from all points of comparison between the solution domains.

The error of the computed fields in Ω_T due to reflections from the ABC's are obtained by subtracting the field at any point within Ω_T at any time step from the field at the corresponding space-time point in Ω_B

$$e(i,j) = E_z^T(i,j) - E_z^B(i,j) \tag{2}$$

where $E_z^T(i,j)$ is the electric field component in the test domain Ω_T and $E_z^B(i,j)$ is the electric field component in the large control domain Ω_B. The error can then be calculated either locally by plotting $e(i,j)$ versus position along a line parallel to the test ABC or globally by computing the sum of the squares of the error at each grid point of Ω_T. The global error is defined to be

$$E = \sum_i \sum_j e^2(i,j). \tag{3}$$

Fig. 1 compares the magnitude of the local error at n =100 time steps for the 2-D TM_z case for the 7th order HOABC, the 2nd order Mur ABC and 4, 8, 12 and 16 cell thick PML regions. As can be seen, the PML ABC provides more reduction in the local error produced by the absorbing boundary as the layer becomes thicker. The PML ABC provides significantly better results when compared to the Mur 2nd order ABC. Also of significance is the fact that the 7th order HOABC performs as well as a 4 layer PML region when looking at the induced local error. However, thicker PML regions provide superior results to the 7th order HOABC.

Fig. 2 compares the global error versus time step for the various ABC's under consideration. The PML ABC produces significantly less error globally than either of the OWWE based ABC's. Also, that error flattens out after the excitation pulse leaves the computational space whereas the global error due to the OWWE ABC's continues to rise as time progresses. Interestingly, after about 800 time steps, the 2nd order Mur ABC begins to outperform the 7th order HOABC. This is an unexpected result which will be explained by looking at the global error in the frequency domain. The improved global error obtained with the PML ABC becomes rather significant when FDTD calculations are run for a large number of time steps where the accumulated global error from the OWWE based ABC's eventually overwhelms the algorithm causing instability in the solution.

It is also interesting to look at what frequencies error is being introduced to the test solution by the ABC. The methodology used previously is used for this numerical simulation except that the excitation is a ramped sinusoid as follows:

$$E_z(50,25,n) = \left[1 - e^{-(n/20)^2}\right]\sin(2\pi f n\Delta t) \tag{4}$$

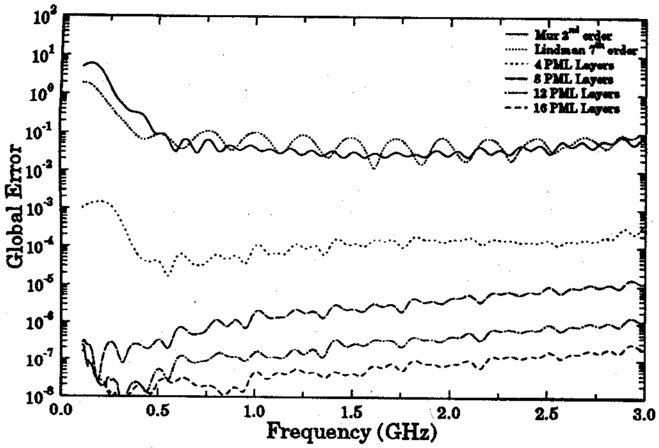

Fig. 3. Frequency spectrum of the induced global error in the test domain by various ABC's after 300 time steps.

where f is the frequency of excitation and the exponential term is used to reduce transients introduced by turning on the source. A solution is run for excitations from 100 MHz to 3 GHz in 10-MHz increments and the global error is calculated after 300 time steps. This point in time is chosen because it is just before the exit of the excitation pulse from the computational space and also because it is near the peak global error found in the previous experiments. The results are shown in Fig. 3. It is interesting to note that the 7th order HOABC gains its advantage over 2nd order Mur at the lower frequencies but there are areas in the higher frequencies where the 2nd order Mur ABC outperforms the 7th order HOABC.

The PML ABC shows excellent results across the band and actually performs better at lower frequencies. This result indicates that the PML ABC will extend the low frequency application of the FDTD algorithm by removing the required electrical distance to the grid truncation widely assumed by OWWE based ABC's. For instance, consider a FDTD solution utilizing a cell size of $\lambda/200$. Using Mur 2nd order ABC's, 200 cells or one wavelength of free space is typically required from the outermost edge of the radiator/scatterer to the grid truncation. Likewise, the HOABC would require one quarter wavelength or 50 cells. A 12 cell deep PML region using a 5-cell deep free space region which contains a near-to-far field transformation surface would require 17 cells from the radiator/scatterer to the grid truncation. This results in a significant savings in computational space required for electrically small cell FDTD solutions.

III. Conclusion

The Berenger PML is a significant advancement in the application of the FDTD method to electromagnetic interactions. The near independence of frequency, the orders of magnitude better accuracy when compared to OWWE based ABC's and the ease of implementation make the Berenger PML an absorbing boundary which will allow the FDTD method to examine problems which previously were difficult if not impossible to analyze. The frequency domain analysis of the various ABC's shows that the PML ABC will make significant contributions at lower frequencies where the FDTD method has been restricted by the electrical distance to the grid truncation required of OWWE based ABC's. As will be seen in a future paper, the PML ABC has been used to accurately predict the wideband input impedance and far-field radiation patterns of a 14′ loop antenna in the HF (2–30 MHz) band utilized on a full scale Apache helicopter. The calculations were made with a cubical cell size of 0.1778 m, electrical distances to the start of a 12 cell PML region as small as $\lambda/150$ at 2 MHz and a solution run of 32 768 time steps.

Another significant result of this study is the realization that HOABC's provide results in middle to upper frequency regions that are no better than Mur 2nd order ABC's. The main advantage of HOABC's over Mur 2nd order ABC's is realized at lower frequencies where electrically small cell sizes are used. The HOABC's allow fewer cells to be applied to free-space from the scatterer or radiator to the grid truncation. A significant advantage of the Berenger PML ABC is that these electrical distance to the absorbing boundary requirements are reduced significantly.

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